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GLOBAL HOLOMORPHIC ONE-FORMS AND THE ALBANESE MAPS ON PROJECTIVE VARIETIES

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§1. Global holomorphic one-forms on projective manifolds with ample canonical bundles

Holomorphic vector fields on compact Kähler manifolds have been used by several different authors to investigate the geometries of the manifolds. In particular, the existence of a holomorphic vector field with non empty zero-locus on a manifold imposes restrictions on the topology of the manifold. For instance, Carrell and Lieberman have shown that *if a Kähler manifold* X *admits a holomorphic vector field* v *whose zero-locus* $Z = Z(v)$ *is not empty, then* $h^{p,q}(X) = dimH^q(X, \Omega_X^p) = 0$ *if* $|p-q| > dimZ$.

In this note, We shall study the zero-locus of global holomorphic one-forms on projective manifolds with ample canonical bundles.

Theorem 1. *Let* X *be a projective manifold with ample canonical bundle* K_X *and* $\omega \in H^0(X, \Omega_X^1)$ *be a holomorphic one-form on* X. *Then the zero-locus* $Z = Z(\omega)$ *of* ω *is not empty.*

Let X be a projective manifold with ample canonical bundle. By the work of Calabi, Aubin, and Yau, X admits a Kähler-Einstein metric and hence the holomorphic cotangent bundle Ω_X^1 is semistable relative to the canonical polarization K_X [Y]. There are many interesting classes of such manifolds. For example, Kawamata [K2] has shown that if X is Kobayashi hyperbolic and is also of general type, then K_X is ample. Other examples include complete intersections in projective spaces with higher degrees, and subvarieties of Abelian varieties with finite stabilizers.

If a compact Kähler manifold X is equipped with a nowhere vanishing holomorphic vector field $v \in H^0(X, \mathcal{T}_X)$, then by Bott's residue formula, the Chern numbers $c_1(X)^n$ and $c_n(X)$ of X vanish. However, there are no analogous results known so far for global holomorphic one-forms. On the other hand, the main ideas used in our proof of Theorem 1 involve splitting the Koszul complex induced by the holomorphic one-form, and the semistabilities of cotangent bundles.

By exactly the same proof, we can generalize Theorem 1 to the following

Theorem 1'. *Let X be a projective manifold and L be a nef line bundle on X such that $K_X + L$ is ample. Let $s \in H^0(X, \Omega^1_X \otimes L)$ be a section. Then the zero-locus $Z = Z(s)$ of s is not empty.*

As an application of Theorem 1, we are able to obtain a result on isotriviality for smooth families of projective manifolds with ample canonical bundles. This result has also been proved by Kovács using a different method [[Kov],Theorem 1].

Theorem 2. *Let $f : X \to A$ be a smooth family of projective manifolds with ample canonical bundle and A is an Abelian variety. Then X is a locally trivial fiber bundle.*

In particular, there exists a finite étale cover $A' \to A$ such that $X \times_A A' \cong X \times A'$ due to the fact that $\mathrm{Aut}(F)$ is finite.

When X is a smooth family of minimal surfaces of general type over an elliptic curve, this had already been shown by Migliorini [Mi]. Our approach to Theorem 2 is base on the fact that if X is not a fiber bundle over A, then K_X must be ample. On the the hand, pullbacks a non-zero one-form from A to X will create a nowhere vanishing one-form on X, this contradicts Theorem 1.

Next, we shall use global holomorphic one-forms to investigate the Euler characteristics on projective manifolds with ample canonical bundles.

Subvarieties of Abelian varieties have been studied intensively during the past two decades ([K1], [U]). In [[U], Theorem 10.3], Ueno gave a beautiful structure theorem of these varieties. He had shown that *if X is a normal* *subvariety of an Abelian variety A, then there exists an étale cover $X' \to X$ such that $X' \cong B \times W$ where B is an Abelian variety, W is a subvariety of an Abelian variety and W is also of general type.* Later, this result has been generalized by Kawamata to varieties which are finite over Abelian varieties [[K1], Theorem 13]. By studing Gauss mappings, Ran [[R], Corollary 2] has shown that *if X is a projective manifold such that the cotangent bundle Ω^1_X is generated by its global sections, then X is of general type if and only if K_X is ample.*

In [[GL], Theorem 1], using deformation theory and generic vanishing theorems, Green and Lazarsfeld proved that : *If X has maximal Albanese dimension, i.e., if the Albanese map $a : X \to Alb(X)$ is generic finite, then $\chi(X, K_X) \geq 0$.* In this paper, in the spirit of Green and Lazarsfeld, we give a characterization of those subvarieties of Abelian varieties which are of general type. Our result was also inspired by the work of J. Kollár [Ko].

Theorem 3. *Let X be a projective manifold such that the cotangent bundle Ω^1_X is generated by its global sections (which equivalent to the Albanese map $X \to Alb(X)$ being an immersion) . Then K_X is ample (equivalent to X being of general type) if an only if $\chi(X, K_X) > 0$.*

Corollary 1. *Let X be a smooth subvariety of an Abelian variety. Then X is of general type if and only if $\chi(X, K_X) > 0$.*

If X is Kähler hyperbolic, then $\chi(X, K_X) > 0$ by Gromov [[G], Theorem 0.4A]. But it is easy to show that if X is a subvariety of an Abelian variety, then in general X is not Kähler hyperbolic. Recently, Ein and Lazarsfeld [[EL], Theorem 3] has proved that if X is birational to a subvariety of an Abelian variety, then X is of general type if and only if $\chi(X, K_X) > 0$.

Combining our Theorem 3 and the generic vanishing theorems of Green and Lazarsfeld [[GL], Theorem 1], we have

Corollary 2. *Let X be a projective manifold such that the cotangent bundle Ω^1_X is generated by its global sections and suppose X is of general type. Let $L \in Pic^0(X)$ be a generic non-torsion (topologically trivial) line bundle. Then $0 < h^0(X, K_X \otimes L) < h^0(X, K_X)$.*

As an immediate consequence of Theorem 1, we can recover the following result of Smyth [[S], Corollary 1].

Corollary 3. *Let X be a projective manifold such that the cotangent bundle Ω_X^1 is generated by its global sections. Then K_X is ample if an only if $c_n(\Omega_X^1) = (-1)^n e(X) > 0$, where $n = dimX$ and $e(X)$ is the topological Euler number.*

We believe that the spannedness of Ω_X^1 is not necessary in Theorem 3. In fact we conjecture that

Conjecture. *Let X be a projective manifold with nef cotangent bundle Ω_X^1. Then K_X is ample if an only if $\chi(X, K_X) > 0$.*

Remark: The above conjecture was inspired by some recent works of Demailly, Peternell and Schneider [[DPS2], Corollary 5.5] on compact Kähler manifolds with nef tangent bundles. They have shown *if X is a compact Kähler manifold with nef tangent bundle, then $-K_X$ is ample (i.e., X is Fano) if and only if $\chi(X, \mathcal{O}_X) > 0$.* Our conjecture can be regarded as a "dual version" of their result.

Unfortunately, so far we are unable to solve our conjecture completely. Nevertheless, using Riemann-Roch, Schur polynomials and the Miyaoka-Yau inequality, we can show

Theorem 4. *Let X be a projective manifold with nef cotangent bundle Ω_X^1. If $\chi(X, K_X) > 0$, then K_X is ample. The converse is true if $dim(X) \leq 4$.*

§2. The Albanese maps on projective manifolds with nef anticanonical bundles

Compact Kähler manifolds with numerically effective anticanonical bundles have been investigated recently by Demailly, Peternell and Schneider [DPS1]. They have shown (among other things): *If X is a compact Kähler manifold with nef $-K_X$. Then $\pi_1(X)$ is a group of subexponential growth.*

One of the most important methods for studying a manifold with nef anticanonical bundle is to analyse the Albanese map associate with the manifold. In [DPS1], Demailly, Peternell and Schneider conjectured that

Conjecture. *Let X be a compact Kähler manifold with $-K_X$ nef. Then the Albanese map $\alpha : X \to A(X)$ is surjective.*

The following result has been proved in [DPS1]:

Theorem. *Let X be a n-dimensional compact Kähler manifold such that $-K_X$ is nef. Then*

(1) *The Albanese map $\alpha : X \to A(X)$ is surjective provided that $dim\alpha(X) = 0,1$ or n, and also $dim\alpha(X) = n - 1$ if X is projective.*
(2) *If X is projective and if the generic fiber F of α has $-K_F$ big, then α is surjective.*

In particular, the Theorem settles the conjecture for projective threefolds. In this paper, our purpose is to show that when X is projective, the conjecture follows from the relative deformation theory which has been developed recently by Kollár, Miyaoka and Mori (see [KMM] and [M]) and by Campana [C].

The precise statements of our results are as follows:

Theorem 5. *Let X be a smooth projective variety over the field of complex numbers $\mathbb{C}$ such that $-K_X$ is nef. Then the Albanese map $\alpha : X \to A(X)$ is surjective and has connected fibers.*

Corollary 4. *Let X be a smooth projective variety over $\mathbb{C}$ such that $-K_X$ is nef. Then we have the irregularity $q(X) = dimH^0(X, \Omega_X) \leq dimX$. Moreover, if the equality holds, then X is isomorphic to an abelian variety.*

More general, we can show the following;

Theorem 6. *Let $\pi : X \to Z$ be a surjective morphism between smooth projective varieties such that $-K_X$ is nef. Then $-K_Z$ is generically nef. In particular, we have the Kodaira dimension $\kappa(Z) \leq 0$. Moreover one of the following holds:*

(1) *Z is uniruled (in particular $\kappa(Z) = -\infty$);*
(2) *There exists a finite étale cover $Y \to Z$ such that $Y \cong A \times V$, where A is an abelian variety and V is a Calabi-Yau manifold, i.e., $K_V =$*

$\mathcal{O}_V$ *and* $\pi_1(V) = \{0\}$. *Moreover if* $\chi(\mathcal{O}_Z) \neq 0$, *then* $|\pi_1(Z)| \leq 2^{\dim(Z)-1}/\chi(\mathcal{O}_Z)$.

The proofs of theorem 5 and 6 are based on the following lemma which is a generalization of a result in (Corollary 2.8, [KMM] and Theorem 2, [M])

Lemma 1. *Let* $\pi : X \to Z$ *be a surjective morphism between smooth projective varieties over* $\mathbb{C}$. *Then there exists no ample divisor* A *on* Z *such that* $-K_{X/Z} - \delta\pi^*A$ *is nef for some* $\delta > 0$.

REFERENCES

[C]. F. Campana, *Connexité rationnelle des variétés de Fano*, Ann. Sci. École Norm. Sup. **4(25)** (1992), 539-545.

[DPS1]Demailly, T. Peternell, M. Schneider, *Kähler manifolds with numerically effective Ricci class*, Compositio Math **89** (1993), 217-240.

[DPS2]Demailly, T. Peternell, M. Schneider, *Compact complex manifolds with numerically effective tangent bundles*, J. Algebraic geometry. **3** (1994), 295-345.

[EL]L. Ein, R. Lazarsfeld, *Singularities of Theta divisors, and the birational geometry of irregular varieties*, preprint (1996).

[GL]M. Green, R. Lazarsfeld, *Deformation theory, generic vanishing theorems and some conjectures of Enriques, Catanses and Beauville*, Invent. Math. **90** (1987), 389-407.

[G]. M. Gromov, *Kahler hyperbolicity and L_2-Hodge theory*, J. Diff. Geom. **33** (1991), 263-292.

[K1]Y. Kawamata, *Characterization of Abelian varieties*, Comp. Math. **43** (1981), 253-276.

[K2]Y. Kawamata, *Minimal model and the Kodaira dimension of algebraic fiber spaces*, J. reine. angew. Math. **363** (1985), 1-46.

[Ko]J. Kollár, *Shafarevich Maps and Automorphic Forms*, Priceton Univ. Press (1995).

[KMM]Kollár, Y. Miyaoka, S. Mori, *Rational connectedness and boundedness of Fano manifolds*, J. Differential Geometry **36** (1992), 765-779.

[Kov]Kovács, *Smooth families over rational and elliptic curves*, preprint (1995).

[M].Y. Miyaoka, *Relative deformations of morphisms and applications to fiber spaces*, Commentarii Math. Univ. Sancti Pauli **42(1)** (1993), 1-7.

[Mi]L. Migliorini, *A smooth families of minimal surfaces of general type over a curve of genus at most one is trivial*, J. Algebraic geometry. **4** (1995), 353-361.

[R]. Z. Ran, *The structure of Gauss-like maps*, Comp. Math. **52** (1984), 171-177.

[S]. B. Smythe, *Weakly ample Kähler manifolds and Euler number*, Math. Ann. **224** (1976), 269-279.

[U]. K. Ueno, *Classification Theory of Algebraic Varieties and Compact Complex Spaces*, Lecture Notes in Mathematics. **439** (1975).

[Y]. S. T. Yau, *Calabi's conjecture and some new results in algebraic geometry*, Proc. Nat. Acad. Sci. USA. **74** (1977), 1798-1799.

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